Compacton and Foldon Excitations in the Generalized (2+1)-dimensional Nizhnik-Novikov-Veselov Equation

Wen-Hua Huang[a,b], Jie-Fang Zhang[b], and Wei-Gang Qiu[a]

[a] School of Science, Zhejiang Huzhou Teachers College, Huzhou 313000, P.R. China
[b] Institute of Nonlinear Physics, Zhejiang Normal University, Jinhua 321004, P. R. China

Reprint requests to W.-H. H.; E-mail: whhuanghz@hutc.zj.cn

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Using the variable separation approach, abundant localized coherent solutions are obtained for the generalized (2+1)-dimensional Nizhnik-Novikov-Veselov (GNNV) equation. Two special types of localized excitations, compactons and foldons, are discussed. The behavior of the interactions for three-compacton solutions and two foldon solutions are investigated, and many interesting interaction properties are revealed. – PACS number: 02.30.Jr, 03.40.Kf, 03.65.Ge,05.45.Yv,03.65.-w

Key words: The GNNV Equation, Compacton, Foldon, The Variable Separation Approach.

1. Introduction

In the study of nonlinear physical models, it is very important to find accurate localized soliton solutions and investigate the interaction of the soliton solutions in the case of (2+1)dimensions. In recent years much effort has been focused on localized soliton solutions for (2+1)-dimensional nonlinear models, and many types of localized excitations, such as solitoffs, dromions, rings, lumps, breathers, instantons, peakons, campactons and localized chaotic and fractal patterns, etc, have been found [1 – 8]. Meanwhile there exist many new studies on the interaction of soliton solutions or localized soliton structures in (2+1)-dimensional physical models [9 – 12].

For the (2+1)-dimensional generalized Nizhnik-Novikov-Veselov (GNNV) equation [13]

$$\begin{aligned} u_t + au_{xxx} + bu_{yyy} + cu_x + du_y \\ = 3a(uv)_x + 3b(uw)_y, \end{aligned} \tag{1}$$

$$u_x = v_y, \;\; u_y = w_x, \tag{2}$$

where $a, b, c,$ and d are arbitrary constants, several types of the soliton solutions and localized excitations have been studied by many authors. For instance, Boiti et al. [13] solved the GNNV equation via the inverse scattering transformation. Radka and Lakshmanan [14] obtained the multi-dromion solutions by means of the bilinear method. Zhang [15] obtained many exact solutions of this system, based on an extended homogeneous balance approach [15]. Localized chaotic and fractal excitation patterns are derived by Zheng et al. [16]. However, the GNNV equation yields many interesting soliton structures that have not yet been found, and the interaction between the solitons is still worth of studying. In this paper we apply the variable separated approach (VSA), which was proposed by Lou [17], to the GNNV equation (1) with (2) and study its special localized compacton and foldon excitations.

2. Variable Separation Solutions of the GNNV Equation

To use VSA, we perform the Bäcklund transformation

$$\begin{aligned} &u = -2(lnf)_{xy} + u_0, \;\; v = -2(lnf)_{xx} + v_0, \\ &w = -2(lnf)_{yy} + w_0, \end{aligned} \tag{3}$$

where $\{u_0, v_0, w_0\}$ is an arbitrary known seed solution of the GNNV equation. For simplicity we consider the special case

$$u_0 = 0, \;\; v_0 = p_0(x,t), \;\; w_0 = q_0(y,t). \tag{4}$$

Substituting (3) with (4) into (1) leads to the following symmetry form

$$(c - 3ap_0)(2ff_xf_{xy} - f^2f_{xxy} - 2f_x^2f_y + ff_{xx}f_y)$$

$$+ (d - 3bq_0)(2ff_yf_{xy} - f^2f_{yyx} - 2f_y^2f_x + ff_{yy}f_x)$$

$$+a[2f_x(2ff_{xxxy}+3f_{xx}f_{xy}-f_{xxx}f_y)-f^2f_{xxxxy}$$
$$+ff_{xxxx}f_y-2ff_{xxx}f_{xy}-6f_x^2f_{xxy}]$$
$$+b[2f_y(2ff_{yyyx}+3f_{yy}f_{xy}-f_{yyy}f_x)-f^2f_{yyyyx}$$
$$+ff_{yyyy}f_x-2ff_{yyy}f_{xy}-6f_y^2f_{yyx}]$$
$$+f(f_{xy}f_t+f_{xt}f_y+f_{yt}f_x)-f^2f_{xyt}-2f_xf_yf_t$$
$$+3(ap_{0x}+bq_{0y})(f^2f_{xy}-ff_xf_y)=0, \quad (5)$$

which can be rewritten as a multi-linear equation with respect to f, while (2) is satisfied identically under the transformation (3) with (4). To get the separation solutions and study the interaction of solitons conveniently, we select f as follow

$$f=p_1(x,t)+p_2(x,t)q(y,t), \quad (6)$$

where $p_1\equiv p_1(x,t) and p_2\equiv p_2(x,t)$ are functions of $\{x,t\}$ and $q\equiv q(y,t)$ is a function of $\{y,t\}$. It is clear that the variables x and y now have been separated totally.

Substituting (6) into (5), we have

$$(2f_x-f\partial_x)[a(p_{1xxx}p_2-p_{2xxx}p_1)$$
$$+3a(p_{2xx}p_{1x}-p_{1xx}p_{2x})+p_2p_{1t}$$
$$-p_1p_{2t}+(3ap_0-c)(p_1p_{2x}-p_2p_{1x})]$$
$$+(p_1p_{2x}-p_2p_{1x})(2p_2-q_y^{-1}f\partial_y)$$
$$\cdot(-bq_{yyy}-q_t-dq_y+3bq_yq_0)=0. \quad (7)$$

Because p_1 and p_2 are y-independent and q is x-independent, (7) can be divided into two equations:

$$p_1p_{2t}-p_2p_{1t}=a(p_{1xxx}p_2-p_{2xxx}p_1)$$
$$+3a(p_{2xx}p_{1x}-p_{1xx}p_{2x}) \quad (8)$$
$$+(3ap_0-c)(p_1p_{2x}-p_2p_{1x}),$$

$$q_t=3bq_yq_0-bq_{yyy}-dq_y. \quad (9)$$

Thanks to the arbitrariness of the functions p_0 and q_0, the soliton solution of the GNNV equation may have quite rich structures. In fact, it is not necessary to solve (8) and (9) because of the arbitrariness of the functions p_0 and q_0. In other words, if we fix the functions p_0 and q_0 as

$$p_0=\frac{1}{3a}\Big\{\big[p_1p_{2t}-p_2p_{1t}-a(p_{1xxx}p_2-p_{2xxx}p_1) \quad (10)$$
$$-3a(p_{2xx}p_{1x}-p_{1xx}p_{2x})\big](p_1p_{2x}-p_2p_{1x})^{-1}+c\Big\},$$

$$q_0=\frac{1}{3bq_y}(q_t+bq_{yyy})+\frac{d}{3b}, \quad (11)$$

then p_1,p_2 and q become three arbitrary functions.

Finally, substituting (6) into (3) we find that the GNNV equation possesses an exact solution:

$$u=-2(lnf)_{xy}$$
$$=-\frac{2p_{2x}q_y}{p_1+p_2q}+\frac{2(p_{1x}+p_{2x}q)p_2q_y}{(p_1+p_2q)^2}, \quad (12)$$

$$v=-2(lnf)_{xx}+v_0$$
$$=-\frac{2p_{1xx}+p_{2xx}q}{p_1+p_2q}+\frac{2(p_{1x}+p_{2x}q)^2}{(p_1+p_2q)^2}+p_0, \quad (13)$$

$$w=-2(lnf)_{yy}+w_0$$
$$=-\frac{2p_2q_{yy}}{p_1+p_2q}+\frac{2p_2^2q_y^2}{(p_1+p_2q)^2}+q_0. \quad (14)$$

From (10)–(14) we notice that for general choices of p_1, p_2 and q there may be some singularities for u, v and w. We have to choose the functions p_1, p_2 and q carefully to avoid the singularities. When the functions $p_1(x,t)$, $p_2(x,t)$ and $q(y,t)$ are selected to avoid the singularities of (12), (13), and (14), (12), (13), and (14) reveal quite abundant soliton structures. For simplicity we only discuss soliton solutions and the interaction properties of the localized excitations of the field u (12) which can be rewritten as

$$u=-2ln(f)_{xy}=\frac{-2q_y(p_{2x}p_1-p_2p_{1x})}{(p_1+p_2q)^2}. \quad (15)$$

Selecting the proper arbitrary functions $p_1(x,t)$, $p_2(x,t)$ and $q(y,t)$, we can easily obtain many localized soliton solutions such as solitoffs, dromions, ring solitons, breathers, instantons, peakons, etc. In the following, for the field u (15) we will study two special types of localized coherent soliton solution, called compacton and foled solitrary wave solutions respectively.

3. The Interaction Among Travelling Compactons

In 1+1 dimensions, there are many papers to study compactons which are completely compacted in a small finite region and are outside of the region, while

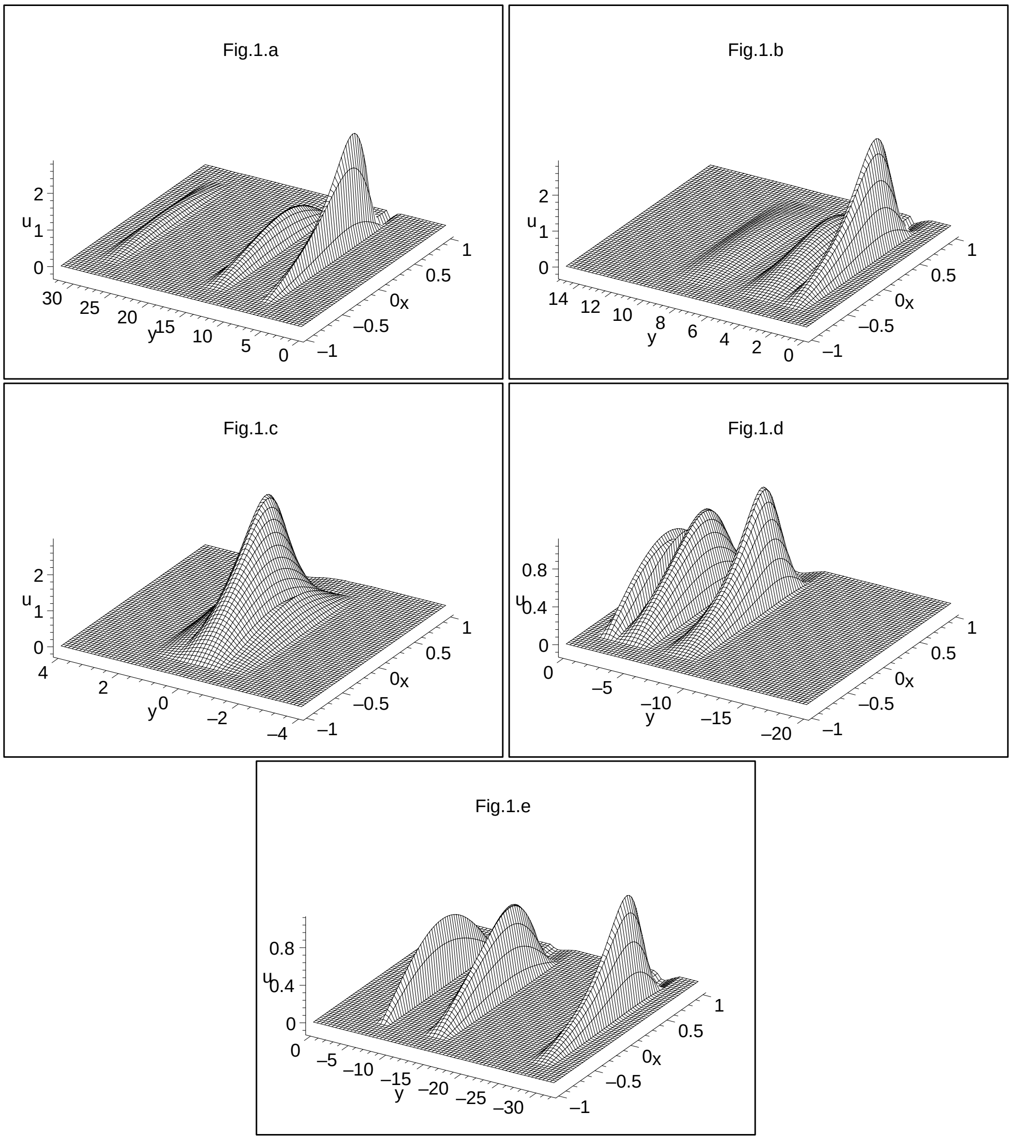

Fig. 1. The evolution of the interaction of three compactons for the field u (15) with (16) – (18) and (19) at the times (a) $t = -14$, (b) $t = -4$, (c) $t = 0$, (d) $t = 4$, (e) $t = 14$.

there are only few papers to investigate higher dimensional compactons [5, 6]. If we select p_1, p_2 and q as

$$p_1 = d_0 + \sum_{i=1}^{N} \begin{cases} 0, \ \text{if} \ l_i(x-x_{1i}) \le -\frac{\pi}{2}, \\ d_i \sin(l_i(x-x_{1i})) + d_i, \\ \quad \text{if} \ -\frac{\pi}{2} < l_i(x-x_{1i}) \le \frac{\pi}{2}, \\ 2d_i, \ \text{if} \ l_i(x-x_{1i}) > \frac{\pi}{2}, \end{cases} \tag{16}$$

$$p_2 = \sum_{i=1}^{n} \begin{cases} 0, \ \text{if} \ K_i(x-x_{2i}) \le -\frac{\pi}{2}, \\ L_i \sin(K_i(x-x_{2i})) + L_i, \\ \quad \text{if} \ -\frac{\pi}{2} < K_i(x-x_{2i}) \le \frac{\pi}{2}, \\ 2L_i, \ \text{if} \ K_i(x-x_{2i}) > \frac{\pi}{2}, \end{cases} \tag{17}$$

$$q = \sum_{i=1}^{M} \begin{cases} 0, \ \text{if} \ k_i y - \omega_i t + y_{0i} \le -\frac{\pi}{2}, \\ b_i \sin(k_i y - \omega_i t + y_{0i}) - b_i, \\ \quad \text{if} \ -\frac{\pi}{2} < k_i y - \omega_i t + y_{0i} \le \frac{\pi}{2}, \\ -2b_i, \ \text{if} \ k_i y - \omega_i t + y_{0i} > \frac{\pi}{2}, \end{cases} \tag{18}$$

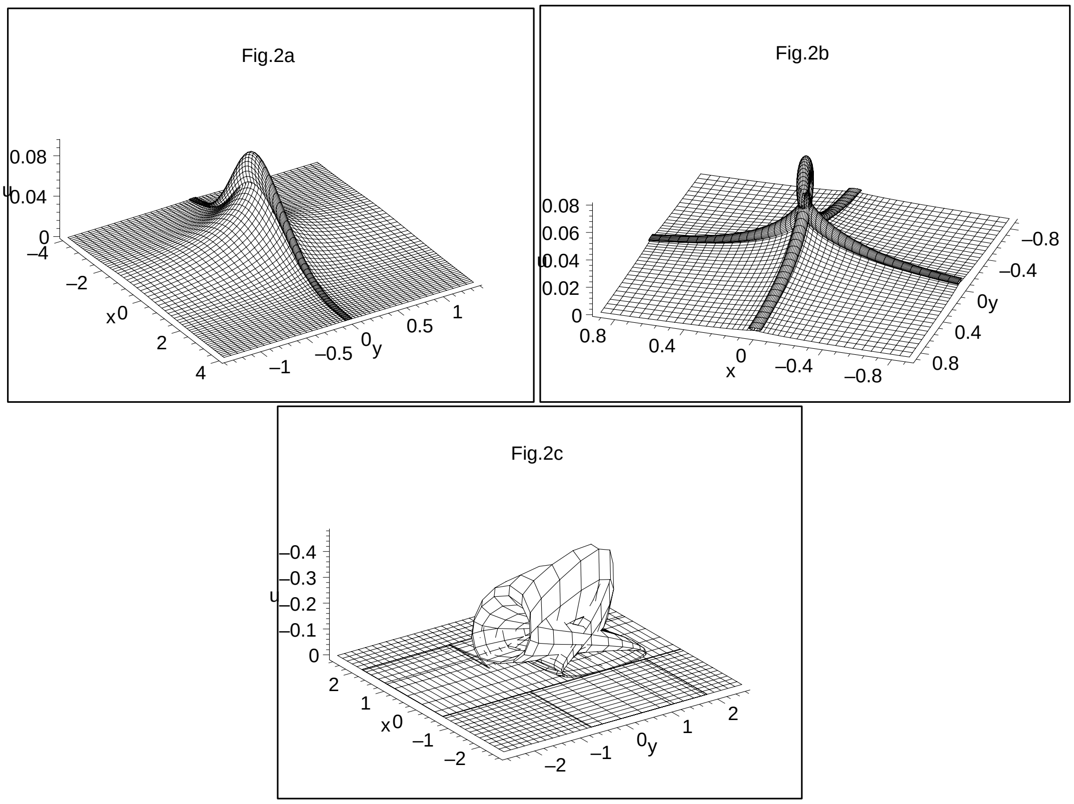

Fig. 2. Three typical folded solitary wave structures for the field u expressed by (15) ant $t=0$ with (14), (15) and the related concrete selections are: (a) $p_{1x}=\mathrm{sech}^2(\xi-v_1t)$, $p_{2x}=0$, $q_y=\mathrm{sech}^2(\eta-v_2t)$, $x=\xi+b\tanh(\xi-v_1t)$, $y=\eta-1.15\tanh(\eta-v_2t)$, $p_1=\int^{\xi}p_{1x}x_{\xi}\mathrm{d}\xi+l_0$, $p_2=1$, $q=\int^{\eta}q_yy_{\eta}\mathrm{d}\eta+h_0$ and $b=1, l_0=5, h_0=0$; (b) same as (a) but with $b=-1.15$; (c) p_{1x}, p_1, q are same as (a), however $p_{2x}=\mathrm{sech}^6(\xi-v_1t)$, $p_2=\int^{\xi}p_{2x}x_{\xi}\mathrm{d}\xi+l_1$, $x=\xi+2\tanh(\xi-v_1t)+\tanh^2(\xi-v_1t)-k_0\tanh^3(\xi-v_1t)$, $y=\eta+2\tanh(\eta-v_2t)+\tanh^2(\eta-v_2t)-k_0\tanh^3(\eta-v_2t)$ and $l_0=5$, $l_1=0$, $h_0=0$, $k_0=5.4$.

where N, n and M are arbitrary integers, and b_i, l_i, k_i, d_i, L_i, K_i, ω_i, x_{1i}, x_{2i} and y_{0i} are arbitrary constants, we can obtain multi-compacton solutions of the (2 + 1)-dimensional GNNV equation. Figure 1 shows the interaction of three compacton solutions of the field u (15) with (16) – (18) and

$$\begin{aligned}&N=n=1,\ M=3,\ d_0=20,\\&l_1=d_1=-k_1=-k_2=\omega_1=\omega_3=b_2=1,\\&b_1=-k_3=\omega_2=K_1=2L_1=2,\ b_3=3/2,\\&x_{11}=x_{21}=y_{01}=y_{02}=y_{03}=0.\end{aligned}\qquad(19)$$

From Fig. 1, we can find the interactions of three compactons are not elastic because they exchange partially their shapes.

4. The Folded Solitary Wave Solutions

In reality, there exist very complicated folded phenomena such as the folded protein [18], folded brain and skin surface, and many other kinds of folded biological systems [19]. The loop solitons are thought as a class of simplest folded waves in (1+1)-dimensional integrable system [20]. Recently, folded solitary waves and foldons were found by Tang and Lou [21] in some (2+1)-dimensional nonlinear models, such as the (2+1)-dimensional dispersive long wave equation, the (2+1)-dimensional Burgers equation, etc. Here we study the folded solitary waves and foldons directly starting from the field u (15) due to the arbitrariness of $p_1(x,t), p_1(x,t)$ and $q(y,t)$. It is considered that these special excitations should be described by multi-valued functions. We first concentrate on how to find some

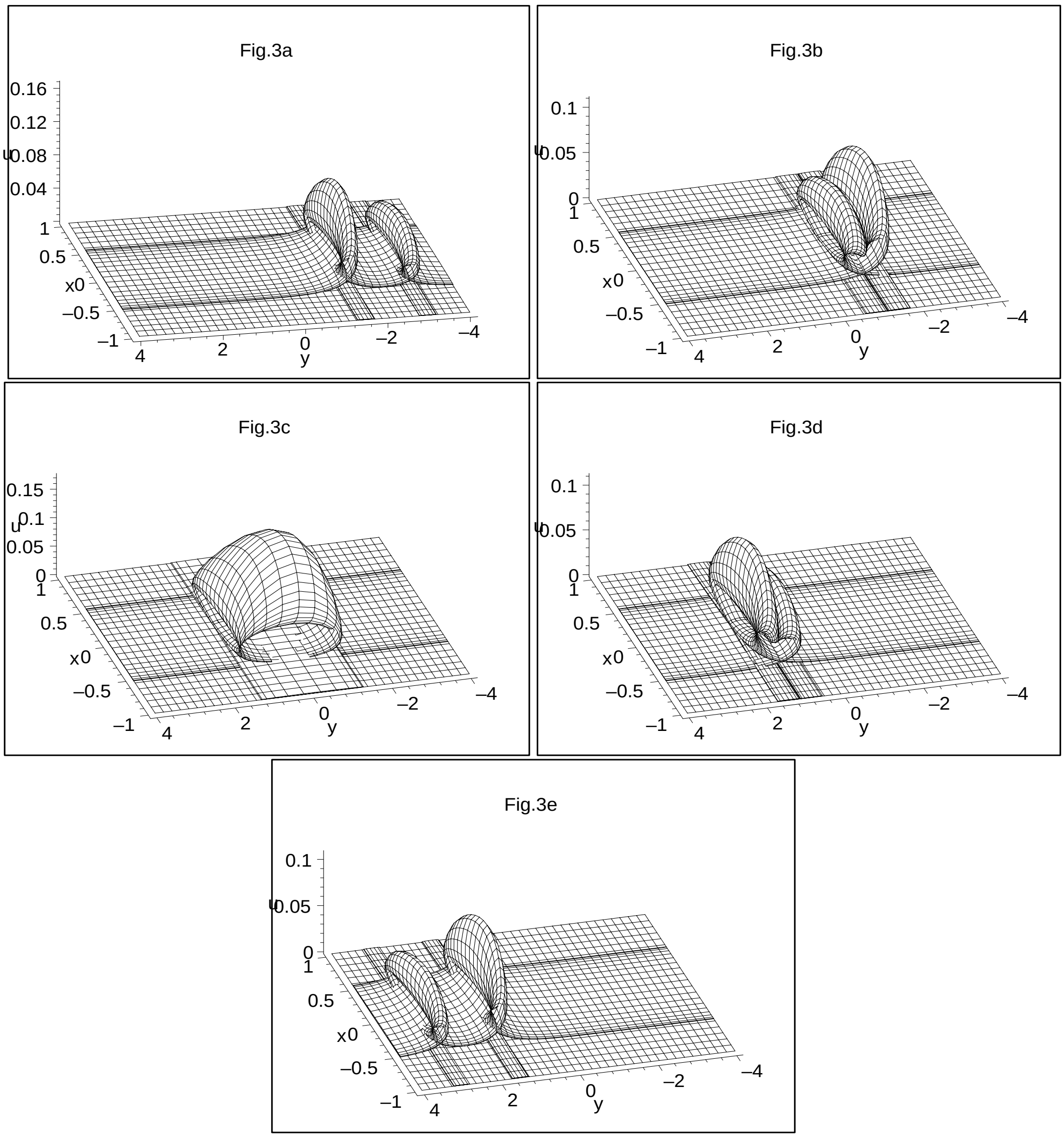

Fig. 3. Evolution plots of two foldon for the field u expressed by (15) with $q_y = 0.8\mathrm{sech}^2\eta + 0.5sech^2(\eta - 0.25t)$, $p_{1x} = sech^2\xi$, $p_{2x} = 0$, $y = \eta - 1.5\tanh\eta - 1.5\tanh(\eta - 0.25t)$, $x = \xi - 2\tanh\xi$, $p_1 = \int^{\xi} p_{1x}x_{\xi}\,\mathrm{d}\xi + 4$, $p_2 = 1$, $q = \int^{\eta} q_y y_{\eta}\,\mathrm{d}\eta$ at times $(a)t = -18$, $(b)t = -9$, $(c)t = 0$, $(d)t = 9$, $(e)t = 18$, respectively.

types of folded solitary waves and foldons of the field u briefly. A localized functions px in the form

$$px = \sum_{j=1}^{M} f_j(\xi + v_j t), \;\; x = \xi + \sum_{i=1}^{M} g_j(\xi + v_j t), \tag{20}$$

where $v_1 < v_2 < \ldots v_M$ are arbitrary constants and $(f_j, g_j), \;\forall_j$ are localized functions with the properties $f_j(\pm\infty), \;\; g_i(\pm\infty) = G_i^{\pm} = constant$. From the second equation of (20) we know that ξ may be a multi-valued function in some possible regions of x by selecting the functions g_j suitably. So the function px may be a multi-valued function of x in these regions though it is a single valued function of ξ. It is also clear that px is an interaction travelling solution of M localized excitations, because of the property $\xi\mid_{x\to\infty}\to\infty$. Now, if we take the arbitrary functions which appear in (15), then we can get various types of folded solitary waves and / or foldons.

In Fig. 2, three typical special folded solitary waves are plotted for the field quantity u shown by (15) with $p_x = px, p = \int^{\xi} p_x x_{\xi} d\xi + l_0$ and the functions q being given in a similar way:

$$q_y = \sum_{j=1}^{M} Q_j(\eta + v_j t),\; y = \eta + R(\eta + v_j t), \\ q = \int^{\eta} q_y y_\eta \mathrm{d}\eta + h_0, \tag{21}$$

where l_0 and h_0 are arbitrary integration constants. In (21), $Q_j(\eta), \forall j$ and $R(\eta)$ are localized functions of η. The more detailed choices of the functions of the figures are given in the figure legends.

Figures 3a – e are plotted to show the possible existence of foldons which are given by (15). The concrete choice of functions is also given in the figure legend. From Fig. 3a and e we can see that the interaction of two foldons is completely elastic. Because one of the velocities of foldons has been chosen as zero, it can also be seen that there are phase shifts for two foldons. Especially, before the interaction the static foldon (the large one) is located at $y = -1.5$, and after the interaction, the large one is shifted to $y = 1.5$.

5. Summary and Discussion

To summarize, using the variable separation approach, we obtain abundant localized coherent solutions for the GNNV equation because of the existence of three arbitrary functions appearing in the seed solution. We investigate the behavior of the interactions for three-compacton solutions and find the interactions may be not completely elastic. We also obtain folded solitary waves and foldons by selecting arbitrary functions appropriately, and find that the foldons may be folded quite freely and complicatedly and possess quite rich structures and interaction behaviors. The explicit phase shifts for the localized structures presented by u (15) have been given. More about the method and whether this phenomena of localized coherent structures for other higher-dimensional processes is further worth studying.

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